

Angular Fulde-Ferrell-Larkin-Ovchinnikov state in cold fermion gases in a toroidal trap

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We study the angular Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state, in which the rotation symmetry is spontaneously broken, in population imbalanced fermion gases near the BCS-BEC crossover. We investigate the superfluid gases at low temperatures on the basis of the Bogoliubov-de Gennes equation, and examine the stability against thermal fluctuations using the T -matrix approach beyond the local-density approximation. We find that the angular FFLO state is stabilized in the gases confined in the toroidal trap but not in the harmonic trap. The angular FFLO state is stable near the BCS-BEC crossover owing to the formation of pseudogap. Spatial dependences of number density and local population imbalance are shown for an experimental test.

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Superfluidity in cold fermion gases provides vast opportunities to study novel quantum phenomena.¹ One of the goals of current studies is the realization of the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state^{2,3} in population imbalanced superfluid gases.^{4,5} FFLO superfluidity/superconductivity is attracting growing interest in a variety of fields, such as condensed-matter physics,⁶ astrophysics, and nuclear physics.⁷ Since many parameters can be experimentally controlled,¹ cold fermion gases are promising candidates for the FFLO state.^{8,9}

Spontaneous breaking of space symmetry is a characteristic feature of the FFLO state. However, no firm evidence has been obtained for the space symmetry breaking in the condensed-matter physics. In contrast to the superconductors, the spatial structure of superfluid is directly measured in cold atom gases. Therefore, it is highly desired to observe the space symmetry breaking due to the FFLO superfluidity in cold fermion gases. While the translation symmetry plays a major role in superconductors, cold atom gases lack translation symmetry owing to the trap potential. Instead, the rotation symmetry is well defined in the latter. The purpose of this Rapid Communication is to investigate the FFLO state which spontaneously breaks the rotation symmetry.

On the basis of the mean-field Bogoliubov-de Gennes (BdG) equations, some authors investigated the radial FFLO (R-FFLO) state^{10,11,12,13,14} in which the order parameter changes its sign along the radial direction around the edge of the harmonic trap. However, no space symmetry is broken in the R-FFLO state, and therefore it is difficult to distinguish it from the phase separated state.^{15,16,17} In this Rapid Communication we show that the angular FFLO (A-FFLO) state with broken rotation symmetry is stabilized in the toroidal trap.¹⁸ Several experiments are proposed for an unambiguous evidence for the A-FFLO state.

The superfluidity has been realized in the imbalanced fermion gases near the BCS-BEC crossover.^{4,5,19,20,21} However, that is not achieved in the BCS limit since

the transition temperature T_c is too small. Because the mean-field theory breaks down near the BCS-BEC crossover,^{1,22} a theoretical treatment beyond the BdG equations is desired for the study of cold fermion gases. To this end, the local-density approximation (LDA) has been used in the literature.^{8,15,16,17} However, a theory beyond the LDA is needed to study the superfluid phase with broken space symmetry. For these theoretical requirements we adopt the real-space self-consistent T -matrix approximation (RSTA).^{23,24} The reliability of the RSTA has been examined in the uniform system by comparing it with the nonperturbative infinite-loop order theory.²⁵ We found that the RSTA is quantitatively valid at least in the BCS side of BCS-BEC crossover.

We here investigate the gases confined in the (quasi-)two-dimensional space. The two-dimensional gas is produced in the pancake potential $\omega_z \gg \omega_\perp$ with ω_z and ω_\perp being the harmonic trap frequency along the axial and radial directions, respectively. The one-dimensional optical lattice along the axial direction also produces the quasi-two-dimensional gas.²⁶ Since the fluctuation completely suppresses the continuous symmetry breaking in one- and two-dimensional systems at finite temperatures, a weak three dimensionality is assumed to realize the state with broken space symmetry. The following calculation is carried out in the two-dimensional model for simplicity, and the singularity of the low-dimensional model is cut off by a phenomenological procedure.

We adopt the lattice Hamiltonian given as

$$H = -t \sum_{\langle \vec{r}, \vec{r}' \rangle, \sigma} c_{\vec{r}, \sigma}^\dagger c_{\vec{r}', \sigma} + \sum_{\vec{r}, \sigma} (V(|\vec{r} - \vec{r}_0|) - \mu_\sigma) n_{\vec{r}, \sigma} + U \sum_{\vec{r}} n_{\vec{r}, 1} n_{\vec{r}, 2}, \quad (1)$$

where $\sigma = 1, 2$ denote two hyperfine states, \vec{r}_0 is the center of the trap, and $n_{\vec{r}, \sigma} = c_{\vec{r}, \sigma}^\dagger c_{\vec{r}, \sigma}$ is the number operator of σ particles. We take the unit $\hbar = c = 1$. The symbol $\langle \vec{r}, \vec{r}' \rangle$ denotes the summation over nearest neighbor sites. The chemical potential μ_σ for σ particles

is determined so that the number of each particle is N_σ . The particle number and the imbalance are expressed as $N = N_1 + N_2$ and $P = (N_1 - N_2)/(N_1 + N_2)$, respectively. The lattice model is adopted for simplicity, but the discreteness of the lattice is negligible since we assume a small particle density $N/N_L = 0.1$, where $N_L = L \times L$ is the number of lattice sites. Therefore, the following results are valid for continuous systems without lattices in the two-dimensional space. Since $N_L = 38 \times 38$ in our calculation, the particle number is $N \sim 144$. We take the unit of length d so that $1/2md^2 = t = 1$, where m is the mass of atoms. We define the Fermi energy as $\varepsilon_F = \mu - \varepsilon_0$, where ε_0 is the energy of the lowest eigenstate and μ is the chemical potential at $P = 0$ and $U = 0$. We find that the superfluidity is the leading instability and no spin/charge density wave occurs in this model.

The last term of Eq. (1) describes the s -wave attractive interaction. We assume $U/t = -5$, which leads to $U/\varepsilon_F = -3.1$. The BCS-BEC crossover is characterized in the two-dimensional system through the two-particle binding energy e_b rather than the three-dimensional scattering length a_s .²⁷ In the uniform system, the binding energy is related with the chemical potential shift as $\Delta\mu = \mu - \mu_0 = -\frac{e_b}{2}$, where μ_0 is the chemical potential shifted by the Hartree term. The order parameter is described as $\Delta_0 = \sqrt{2e_b\varepsilon_F}$. Our calculations of $\Delta\mu$ and Δ_0 consistently lead to $e_b \sim 0.43\varepsilon_F$ for $U/t = -5$. Since the BCS-BEC crossover occurs around $e_b \sim \varepsilon_F$, our model is close to the BCS-BEC crossover slightly in the BCS side. The binding energy e_b and effective interaction U are related with the three-dimensional scattering length a_s through the confinement length $a_z = 1/\sqrt{m\omega_z}$.²⁸

The trap potential is assumed to be $V(r) = \frac{1}{2}\omega_{ho}(r/r_0)^2 + \omega_{tr}\exp(-r/\xi)$. This potential describes the harmonic trap for $\omega_{tr} = 0$ and the toroidal trap for $\omega_{tr} \neq 0$. We found that the A-FFLO state is unstable in the harmonic trap in the whole parameter range. Therefore, we here show the results for the toroidally trapped system with $\omega_{ho} = 12$, $\omega_{tr} = 8$, and $\xi = 5$. The A-FFLO state is stabilized for any $\omega_{tr}/\omega_{ho} > 0$, whose reason will be discussed later.

We first analyze the model within the mean-field BdG equation and later investigate the role of thermal fluctuations on the basis of the RSTA. We obtain the mean-field Hamiltonian of BdG equations as $H = -t \sum_{\langle \vec{r}, \vec{r}' \rangle, \sigma} c_{\vec{r}, \sigma}^\dagger c_{\vec{r}', \sigma} + \sum_{\vec{r}, \sigma} W_\sigma(\vec{r}) n_{\vec{r}, \sigma} - \sum_{\vec{r}} [\Delta(\vec{r}) c_{\vec{r}, 1}^\dagger c_{\vec{r}, 2}^\dagger + \text{c.c.}]$, where $W_\sigma(\vec{r}) = V(|\vec{r} - \vec{r}_0|) + U n_{\bar{\sigma}}(\vec{r}) - \mu_\sigma$, $\bar{\sigma} = 3 - \sigma$, $n_\sigma(\vec{r}) = \langle n_{\vec{r}, \sigma} \rangle$, and $\Delta(\vec{r}) = U \langle c_{\vec{r}, 1} c_{\vec{r}, 2} \rangle$. The unphysical ultra-violet divergence in $\Delta(\vec{r})$ ^{1,27} is naturally cut off since we adopt the lattice model. We numerically determine the stable phase by comparing the free energy of self-consistent solutions for $n_\sigma(\vec{r})$ and $\Delta(\vec{r})$.

The RSTA has been formulated for the inhomogeneous superconductors.^{23,24} The Green's function $G^\sigma(\vec{r}, \vec{r}', \omega_n)$ and the T -matrix $T(\vec{r}, \vec{r}')$ are obtained by the following

self-consistent equations:

$$G^\sigma(\vec{r}, \vec{r}', \omega_n) = G_0^\sigma(\vec{r}, \vec{r}', \omega_n) + \sum_{\vec{r}_2, \vec{r}_3} G_0^\sigma(\vec{r}, \vec{r}_2, \omega_n) \Sigma^\sigma(\vec{r}_2, \vec{r}_3, \omega_n) G^\sigma(\vec{r}_3, \vec{r}', \omega_n), \quad (2)$$

$$\Sigma^\sigma(\vec{r}, \vec{r}', \omega_n) = U n_{\bar{\sigma}}(\vec{r}) \delta_{\vec{r}, \vec{r}'} - TU^2 T(\vec{r}, \vec{r}') G^{\bar{\sigma}}(\vec{r}', \vec{r}, -\omega_n), \quad (3)$$

$$T(\vec{r}, \vec{r}') = T_0(\vec{r}, \vec{r}') - \sum_{\vec{r}_2} UT_0(\vec{r}, \vec{r}_2) T(\vec{r}_2, \vec{r}'), \quad (4)$$

$$T_0(\vec{r}, \vec{r}') = T \sum_n G^1(\vec{r}, \vec{r}', \omega_n) G^2(\vec{r}, \vec{r}', -\omega_n), \quad (5)$$

where $G_0^\sigma(\vec{r}, \vec{r}', \omega_n)$ is the Green's function for $U = 0$, $\omega_n = (2n + 1)\pi T$ is the Matsubara frequency, and T is the temperature. The thermal fluctuation, which is neglected in the BdG equations, is taken into account in the self-energy $\Sigma^\sigma(\vec{r}, \vec{r}', \omega_n)$ in the one-loop order. The pseudogap in the single-particle excitation as well as the shift in chemical potential are taken into account in the RSTA. The former is neglected in the often-used Nozieres and Schmitt-Rink theory.²⁹ We show that the pseudogap plays an essential role for the stability of A-FFLO state.

The quantum fluctuation is ignored in the RSTA, and therefore the RSTA is valid at finite temperature around T_c .²⁵ Therefore, the RSTA is used to determine the instability to the superfluid state. The T_c is determined by the Thouless criterion. The maximum eigenvalue of $|U|T_0(\vec{r}, \vec{r}')$, namely, λ_L , is unity at $T = T_c$. Since the true long-range order does not occur in finite systems, we adopt the criterion $\lambda_L = 1 - \delta_L = 0.98$ for T_c below which the long-range coherence develops. The singularity due to the one or two dimensionalities is also cut off by this procedure. The following results are not qualitatively altered by the choice of δ_L . This means that the following results are not sensitive to the three dimensionality which is phenomenologically taken into account by a finite δ_L . The density of states (DOS) for particles σ is obtained as $\rho^\sigma(\omega) = -\frac{1}{\pi N_L} \sum_{\vec{r}} \text{Im} G^{\sigma R}(\vec{r}, \vec{r}, \omega)$, where $G^{\sigma R}(\vec{r}, \vec{r}, \omega)$ is the retarded Green's function. The total DOS is expressed as $\rho(\omega) = \rho^1(\omega) + \rho^2(\omega)$.

We first discuss the results of BdG equations. Although the quantitatively reliable result is not obtained near the BCS-BEC crossover, the properties of each phase, such as the local population imbalance, are captured by the BdG equations. We here study the spatial structure of several superfluid phases. Figure 1 shows the phase diagram in which BCS state, R-FFLO state, and A-FFLO state are stabilized. Since the BCS state smoothly changes to the R-FFLO state without any phase transition, we show the crossover line above which the superfluid order parameter changes its sign around the trap edge. With an increase in the population imbalance, the second-order phase transition occurs from the R-FFLO state to the A-FFLO state. Although the transition temperature is significantly overestimated in BdG equations, the successive phase transitions from the BCS

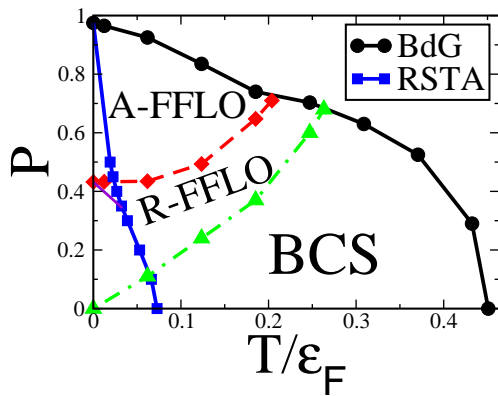


FIG. 1: Phase diagram for the imbalance P and the reduced temperature T/ε_F . Phase boundaries obtained by the BdG equations are shown by circles, triangles, and diamonds. The BCS state, R-FFLO state, and A-FFLO state are shown in the figure. The phase diagram determined by the RSTA is shown by the squares and (purple) thin solid line. The A-FFLO state is stable above the thin solid line, while the R-FFLO or BCS state is stable below it. We fix the particle density $N/N_L = 0.1$ in all figures.

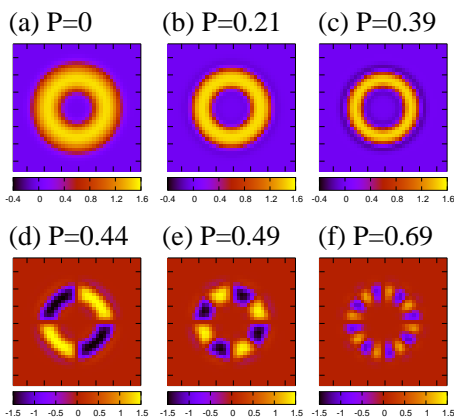


FIG. 2: Spatial dependence of superfluid order parameter $\Delta(\vec{r})$ at $T/\varepsilon_F = 0.00062$. (a) $P = 0$, (b) 0.21, (c) 0.39, (d) 0.44, (e) 0.49, and (f) 0.69, respectively.

state to the A-FFLO state are not altered by the fluctuations, as will be shown on the basis of the RSTA.

We clarify the spatial structures of each phase in Figs. 2-4. First we show the order parameter for various imbalances in Fig. 2. Figure 2(a) shows the BCS state realized in the balanced gas. Figures 2(b) and 2(c) show the R-FFLO state in the imbalanced gases. The rotation symmetry is preserved in these states. As the imbalance is increased in these states, the width of superfluid region shrinks. This deformation is regarded as the *self-one-dimensionalization* of the superfluid along the angular direction and leads to the A-FFLO state for $P > 0.43$. Figures 2(d)-2(f) show the spontaneous rotation symmetry breaking in the A-FFLO state. Thus, the A-FFLO state is an analog of the (quasi)-one-dimensional FFLO

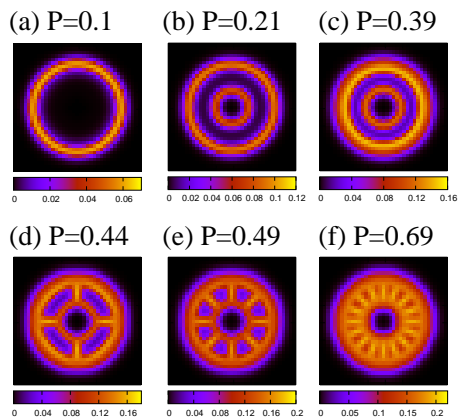


FIG. 3: Spatial dependence of local population imbalance $n_1(\vec{r}) - n_2(\vec{r})$. We assume $P = 0.1$ in (a). The other parameters are the same as in Fig. 2.

state.^{30,31,32,33,34,35,36} It is known that the FFLO state is favored in the quasi-one-dimensional system because of the nesting of Fermi surface.⁶ An important finding of this Rapid Communication is the spontaneous formation of quasi-one-dimensional superfluid in the toroidal trap without any fine tuning. This should be contrasted to the harmonic trap, in which the uniform one-dimensional superfluid is hardly produced. We found that the self-one-dimensionalization occurs and the A-FFLO state is stabilized for any value of $\omega_{tr}/\omega_{ho} > 0$. As the particle density decreases, the quasi-one-dimensional structure is enhanced, and therefore the A-FFLO state is favored. A similar spatial structure has been discussed within the purely one-dimensional model; however, the self-one-dimensionalization has not been noticed.³⁶ Although we investigate the gases in the (quasi)-two-dimensional trap for simplicity, the A-FFLO state will be stabilized by the self-one-dimensionalization in a more general three-dimensional trap too.

Figure 3 shows the spatial dependence of local population imbalance $n_1(\vec{r}) - n_2(\vec{r})$. While the population imbalance appears around the outer and/or inner edges in the R-FFLO state [Figs. 3(a)-3(c)], the spontaneous rotation symmetry breaking is clearly shown in the A-FFLO state [Figs. 3(d)-3(f)]. A clear four-fold anisotropy is shown in Fig. 3(d), while the spatial dependence is smeared with an increase in the imbalance, as shown in Fig. 3(f). Thus, the features of the A-FFLO state are pronounced near the phase boundary to the R-FFLO state. A characteristic feature of A-FFLO state also appears in the particle density $n_1(\vec{r}) + n_2(\vec{r})$ as shown in Figs. 4(d)-4(f). The particle density decreases around the spatial nodes to gain the condensation energy. Owing to the spontaneous symmetry breaking, many A-FFLO states with different nodal directions are essentially degenerate. This degeneracy is slightly lifted by the lattice in our calculation. We show the most stable states in Figs. 2-4.

We here turn to the results of RSTA, and discuss the

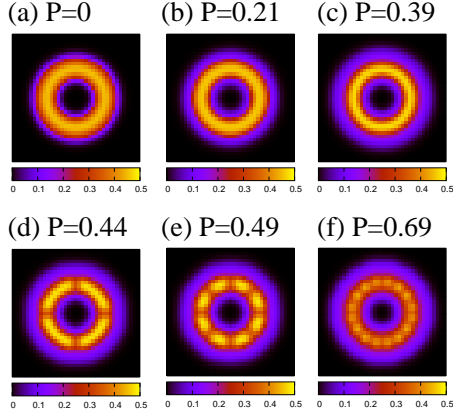


FIG. 4: Spatial dependence of particle density $n_1(\vec{r}) + n_2(\vec{r})$. The parameters are the same as in Fig. 2.

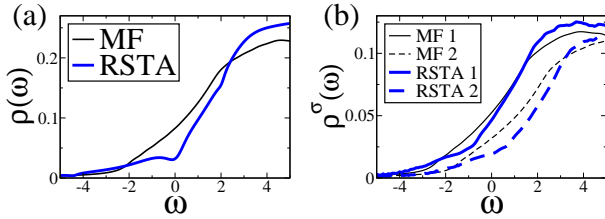


FIG. 5: (a) Total DOS $\rho(\omega)$ in the balanced gas ($P = 0$) at $T/\varepsilon_F = 0.074$. (b) Spin resolved DOS $\rho^\sigma(\omega)$ in the imbalanced gas ($P = 0.4$) at $T/\varepsilon_F = 0.028$. Solid and dashed lines show the DOS for $\sigma = 1$ and 2 , respectively. Thick lines show the results of RSTA, while thin lines are obtained by the mean-field theory for the normal fluid with $\Delta(\vec{r}) = 0$.

roles of the thermal fluctuation. The phase diagram is shown in Fig. 1 and is compared to the mean-field BdG equations. According to the RSTA, the phase transition to the A-FFLO state occurs above $P = 0.35$, although the T_c is decreased by the fluctuation. This means that the A-FFLO state is stable against the fluctuation, in contrast to the previous studies.^{37,38} We see the stability of the A-FFLO state because the phase diagram is plotted for the imbalance P , but not for the “magnetic field” $\mu_1 - \mu_2$. The A-FFLO state seems to be suppressed by the fluctuation when the phase diagram is plotted for the “magnetic field” as in the studies of superconductors.⁶ The relation between the “magnetic field” $\mu_1 - \mu_2$ and the imbalance P is affected by the pseudogap. In Fig. 5 we see the decrease in DOS around $\omega = 0$, namely, the pseudogap. The “magnetic field” for a fixed imbalance is increased by the pseudogap since the “spin susceptibility” $\chi = P/(\mu_1 - \mu_2)$ decreases. The large “magnetic field” leads to the large splitting of Fermi surfaces for particles 1 and 2 and stabilizes the A-FFLO state. In other words, the A-FFLO state is stable near the BCS-BEC crossover

in cold fermion gases since the spin-diffusion time is long enough to conserve the imbalance. This should be contrasted to the superconductors in which the magnetization is not conserved.

We here comment on the superfluid state in the BCS and BEC regimes. In the BCS regime the phase diagram obtained by the BdG equation (Fig. 1) is reliable although the T_c and Chandrasekhar-Clogston limit are decreased. On the other hand, the imbalanced gas in the BEC limit is described by the mixture of molecular bosons and remaining fermions, and then, the FFLO state is not stabilized.

In this Rapid Communication we focused on the superfluid with broken rotation symmetry. Such a spontaneous symmetry breaking is not allowed in the purely one- or two-dimensional systems^{37,38} but is expected to be realized in the weakly three-dimensional system. On the other hand, it is also interesting to investigate the gases in the two-dimensional toroidal trap. Then, the rotation symmetry breaking is suppressed by the gapless collective mode in the isotropic trap but is produced in the anisotropic trap with $\omega_x \neq \omega_y$. Such a giant response to the trap anisotropy may manifest the tendency to the rotation symmetry breaking as well as the singularity of low-dimensional systems.

In summary, we found that the A-FFLO state is stabilized in the population imbalanced fermion gases confined in the toroidal trap. The formation of the R-FFLO state leads to the self-one-dimensionalization of the superfluid and stabilizes the A-FFLO state in the highly imbalanced gases. Then, the rotation symmetry is spontaneously broken. The search for the FFLO state in cold fermion gases has been fruitless probably because the experiments were carried out for the harmonically trapped gases. It is difficult to detect the FFLO state in the harmonic trap since no space symmetry breaking occurs. We suggest that the experiment in the toroidal trap will realize the FFLO state with broken rotation symmetry and will obtain the unambiguous evidence for the FFLO state which has been searched for more than 40 years after the theoretical predictions.^{2,3}

Recently we have become aware of the paper by Chen *et al.* in which a superfluid state similar to the A-FFLO state was investigated in the optical lattice.³⁹ However, the rotation symmetry is not well-defined in the optical lattice. The superfluidity near the BCS-BEC crossover has not been investigated.

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- ¹ S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **80**, 1215 (2008).
 - ² P. Fulde and R. A. Ferrell, *Phys. Rev.* **135**, A550 (1964).
 - ³ A. I. Larkin and Yu. N. Ovchinnikov, *Zh. Eksp. Teor. Fiz.* **47**, 1136 (1964).
 - ⁴ G. B. Partridge *et al.*, *Science* **311**, 503 (2006).
 - ⁵ M. W. Zwierlein *et al.*, *Science* **311**, 492 (2006).
 - ⁶ Y. Matsuda and H. Shimahara, *J. Phys. Soc. Jpn.* **76**, 051005 (2007).
 - ⁷ R. Casalbuoni and G. Nardulli, *Rev. Mod. Phys.* **76**, 263 (2004).
 - ⁸ D. E. Sheehy and L. Radzihovsky, *Phys. Rev. Lett.* **96**, 060401 (2006).
 - ⁹ N. Yoshida and S.-K. Yip, *Phys. Rev. A* **75**, 063601 (2007).
 - ¹⁰ P. Castorina *et al.*, *Phys. Rev. A* **72**, 025601 (2005).
 - ¹¹ J. Kinnunen, L. M. Jensen, and P. Törmä, *Phys. Rev. Lett.* **96**, 110403 (2006).
 - ¹² T. Mizushima, M. Ichioka, and K. Machida, *J. Phys. Soc. Jpn.* **76**, 104006 (2007).
 - ¹³ K. Machida, T. Mizushima, and M. Ichioka, *Phys. Rev. Lett.* **97**, 120407 (2006).
 - ¹⁴ X.-J. Liu, H. Hu, and P. D. Drummond, *Phys. Rev. A* **75**, 023614 (2007).
 - ¹⁵ M. Parish *et al.*, *Nat. Phys.* **3**, 124 (2007).
 - ¹⁶ W. Yi and L.-M. Duan, *Phys. Rev. A* **73**, 031604(R) (2006).
 - ¹⁷ M. Haque and H. T. C. Stoof, *Phys. Rev. A* **74**, 011602(R) (2006).
 - ¹⁸ C. Ryu *et al.*, *Phys. Rev. Lett.* **99**, 260401 (2007).
 - ¹⁹ G. B. Partridge *et al.*, *Phys. Rev. Lett.* **97**, 190407 (2006).
 - ²⁰ M. W. Zwierlein *et al.*, *Nature (London)* **442**, 54 (2006).
 - ²¹ Y. Shin *et al.*, *Phys. Rev. Lett.* **97**, 030401 (2006).
 - ²² Q. Chen *et al.*, *Phys. Rep.* **412**, 1 (2005).
 - ²³ Y. Yanase, *J. Phys. Soc. Jpn.* **75**, 124715 (2006).
 - ²⁴ Y. Yanase and N. Yoroizu, *J. Phys. Soc. Jpn.* **78**, 034715 (2009).
 - ²⁵ Y. Yanase, *J. Phys. Soc. Jpn.* **73**, 1000 (2004).
 - ²⁶ Z. Hadzibabic *et al.*, *Nature* **441**, 1118 (2006).
 - ²⁷ M. Randeria, J.-M. Duan, and L.-Y. Shieh, *Phys. Rev. B* **41**, 327 (1990).
 - ²⁸ J. P. Kestner and L.-M. Duan, *Phys. Rev. A* **76**, 063610 (2007).
 - ²⁹ P. Nozieres and S. Schmitt-Rink, *J. Low Temp. Phys.* **59**, 195 (1985).
 - ³⁰ K. Machida and H. Nakanishi, *Phys. Rev. B* **30**, 122 (1984).
 - ³¹ G. Orso, *Phys. Rev. Lett.* **98**, 070402 (2007).
 - ³² M. Tezuka and M. Ueda, *Phys. Rev. Lett.* **100**, 110403 (2008).
 - ³³ M. M. Parish *et al.*, *Phys. Rev. Lett.* **99**, 250403 (2007).
 - ³⁴ A. E. Feiguin and F. Heidrich-Meisner, *Phys. Rev. B* **76**, 220508(R) (2007); *Phys. Rev. Lett.* **102**, 076403 (2009).
 - ³⁵ H. Hu, X.-J. Liu, and P. D. Drummond, *Phys. Rev. Lett.* **98**, 070403 (2007).
 - ³⁶ F. Ye *et al.*, *J. Phys.: Condens. Matter* **21**, 355701 (2009).
 - ³⁷ H. Shimahara, *J. Phys. Soc. Jpn.* **67**, 1872 (1998).
 - ³⁸ Y. Ohashi, *J. Phys. Soc. Jpn.* **71**, 2625 (2002).
 - ³⁹ Y. Chen *et al.*, *Phys. Rev. B* **79**, 054512 (2009).